arrive at the remarkable result that *all such groups are the direct products of the dihedral rotation group of order* $4p$ *and an abelian group of order* 2^α *and of type* $(1, 1, 1, \ldots)$. When $p>2$ such a group is also the direct product of the dihedral rotation group of order $2p$ and an abelian group of order $2^{\alpha-1}$ and of type $(1, 1, 1, \ldots)$. We proceed to prove these results.

It has been proved above that K is the abelian group of order 2^β and of type $(1, 1, 1, \ldots)$ whenever G contains a minimum number of operators whose orders exceed two. As this must be true of every K for the different non-invariant operators of order two contained in G and as the index of all of these K's is p, it follows that each non-invariant operator of order two contained in G has just p conjugates under G. Since any K is maximal it follows that the operators which are common to two K's are invariant under G, and vice versa. We shall soon see that these common operators include just half of all the operators of G.

As all the products obtained by multiplying t into operators whose orders exceed two are of order two, t must transform every operator whose order exceeds two into its inverse. Hence every non-invariant operator of order two contained in G must have this property. These operators constitute half the operators of $G-K$ and at least half the operators of K. Since they could not constitute more than half of the operators of G (for there are just as many operators that are commutative with any operator in G as there are of those which transform this operator into its inverse) it results that just half the operators of K are invariant under G. These operators constitute an invariant subgroup K' of order $g/2p$.

When $p>2$, G includes an invariant subgroup of order $2p$, since an operator of order p is transformed either into itself or into its inverse by all the operators of G. In this case G is the direct product of K' and this subgroup, the dihedral rotation group of order $2p$, since these two invariant subgroups have only the identity in common and the product of their orders is g*. When $p=2$; that is, when $g=2^m$, K is invariant.† Half of the operators of $G-K$ are of order 4. Hence G includes the octic group as an invariant subgroup. As this has two operators in common with K' such a G is the direct product of the octic group and any subgroup of half the order of K' and included in K' which does not include the commutator of order 2. This completes the proof of the theorems in question; and, as these direct products are so elementary, it seems unnecessary to go into further details.

*Cf. Burnside, Theory of Groups of Finite Order, 1897, p. 44.

†In all other cases K has p conjugates under G.

ON THE MOTION OF A BALL ON A BILLIARD TABLE.

By N. J. LENNES, University of Chicago.

A ball, of radius r, is assumed to move in straight lines and without friction on a rectangular billiard-table, the lengths of whose sides are a and b. Under the assumption that the angle of reflection is equal to the angle of incidence, it is required to discuss completely the motion of the center of the ball when started from a given point and in a given direction.

Under the conditions of the problem the center of the ball must always remain at least at a distance r from the sides of the table, *i. e.*, the center cannot move outside a rectangle whose sides are $a-2r$ and $b-2r$. The problem is obviously equivalent to the problem in which a ball of zero radius is considered to move on a table of dimensions $a-2r$ and $b-2r$. Denote $a-2r$ and $b-2r$ by m and n, and denote the corners of this last rectangle in their cyclic order by A, B, C, D. When we say that the ball strikes one of the sides of the table, we mean that the center of the ball is on one of the segments AB, BC, CD, DA. When the ball strikes a corner of the table the center is at A, B, C, or D. The angle at which the ball strikes the side of the table is the angle between the line of motion and the perpendicular to that side.

The following propositions are immediate consequences of the conditions of the problem:

(1). If the ball is reflected back and forth several times between two opposite sides without striking the remaining sides then the points at which it strikes one side are equidistant and all the lines of motion from either one of these sides to the other are parallel.

(2). If the ball is reflected from one side, as BC, to an adjacent side, as CD, the line of reflection from the side CD is parallel to the line of incidence with BC.

(3). If the ball strikes a corner it is reflected back along the line on which it approached the corner.

(4). If the ball is reflected from a side, as BC, to an adjacent side CD, then it strikes the side CD the same distance from the point C that it would if the table were so extended that the ball would move freely in the direction in which it moved before it struck BC until it should meet the side CD extended.

It follows directly from (3) that if the ball strikes a corner it will retrace its path. These considerations enable us to investigate the periodicity of the motion by considering the problem in the following simplified form, viz., we consider a table in which one side is removed and the table extended indefinitely in that direction. Let the side removed be one whose length is m. On the two extended sides lay off the distance n an indefinite number of times beginning at the corners of the original table. These points are referred to as multiple points of n. The motion of the ball on the original table may now be regarded as transformed into the motion of the ball on the extended table. It is quite easy

to see, from (1)—(4), that if the ball strikes one of the multiple points of n on the extended table then in the corresponding motion on the original table it would strike a corner, and conversely if the ball strikes a corner on the original table then it strikes a multiple point of n on the extended table.

Suppose first that the ball is started from the corner A in a direction whose angle with AB is α. Let the side CD opposite AB be removed and the table extended indefinitely in that direction. Under what conditions will the ball strike a multiple point of n? The answer is: If $m\tan\alpha$ and n are commensurable, the length of AB being m and that of BC being n. If $m\tan\alpha$ and n have a common measure l, which is contained h times in $m\tan\alpha$ and k times in n, then $km\tan\alpha = hn$. Consequently the ball will strike a multiple point of n at a distance hn from the side AB. We may assume that l is the greatest common measure of $m\tan\alpha$ and n. Hence h and k are integers prime to each other. It follows that hn will be the first multiple point of n which the ball strikes. If k is an even number this multiple point of n will be on the side AD extended (the side from which the ball was started).

In every case when the ball strikes the first multiple point of n on the side AD extended, h is odd; for h and k being prime to each other cannot both be even. By direct consideration of the phenomenon on the table this appears from the fact that if h were even we should have by the symmetry of the motion that the ball would strike an odd multiple of n on the side BC half way between the side AB and the multiple point of n in question. This corresponds to the fact that on the original table the ball cannot return to the point A without returning along the path on which it left A; that it cannot strike A without first striking another corner. If k is odd the first multiple point of n reached is on the side BC extended. If h and k are both odd, then the ball will strike the side BC extended in an odd multiple point of n, which means that in the original motion it would strike the corner C. If k is odd and h even the ball will strike the side BC in an even multiple point of n, which corresponds to the point n on the original table.

If k is even the ball will strike each of the sides AD and BC $\frac{1}{2}k$ times in a half period (counting the starting point A but not the point D). If k is odd, it strikes each of the sides AD and BC $\frac{1}{2}(k+1)$ times (in this case counting both the starting point and the point of final impact). Similarly, if h is even the ball will strike each of the sides AB and CD $\frac{1}{2}h$ times counting the starting point but not the point of final impact, and if h is odd it strikes each of the sides AB and CD $\frac{1}{2}(h+1)$ times (counting both the point of departure and the point of final impact).

If k is even and h is odd the first half period ends at D, if h is even and k is odd it ends at B, and if both h and k are odd it ends at C.

If the ball is started from a point P, on the side AD, P being different from A, there are four possibilities, viz., (a) $m\tan\alpha$ is commensurable with n but not with AP or PD. (b) $m\tan\alpha$ is commensurable with n and with either AP or PD (in this case it follows as a theorem that $m\tan\alpha$ is commensurable

with both AP and PD). (c) $m\tan\alpha$ incommensurable with n but commensurable with one of the segments AP and PD (in this case it is a theorem that $m\tan\alpha$ is incommensurable with one of the segments AP and PD). (d) $m\tan\alpha$ incommensurable with n and with both of the segments AP and PD. We consider each case separately.

(a) If $m\tan\alpha$ is commensurable with n but not with AP or PD, then obviously the ball cannot strike a multiple point of n and hence not a corner on the original table but will strike points whose distance from a multiple point of n is AP. If h and k are integers such that $km\tan\alpha = hn$, and if k is even the ball will strike the side AD at a point whose distance from A is $AP+hn$. (We denote by P' all points whose distance from A is $AP+In$ where I is any integer). This corresponds to striking the point P on the original table. Since the ball cannot reach P along the line on which it was started from P but must reach it in a direction whose angle with the side AD is equal to that at which it left P it follows that the ball completes a period on reaching P.

If k is odd the ball will strike a point P' on the side AD which again corresponds to striking P on the closed table.

There is no half period. The number of impacts against each of the sides AD and BC is $\frac{1}{2}k$ if k is even (counting the point P once), and if k is odd (again counting the P once). Similarly, it strikes each of the sides AB and CD $\frac{1}{2}h$ times if k is even, and h times if k is odd.

(b) Under (b) two cases are possible, viz., (1) The ball may strike a point P' on the side AD before it strikes a multiple point of n. The result is the same as in (a). (2) If the ball strikes a multiple point of n before it strikes a point P' it comes under the case discussed above where the ball is started from a corner.

(c) In case $m\tan\alpha$ if incommensurable with n but commensurable with one of the segments AP and PD, then obviously the ball cannot reach a point P'. Further, it can reach at most one corner on the original table; consequently the motion is not periodic in this case.

(d) In case $m\tan\alpha$ is incommensurable with n and both segments AP and PD the motion is obviously not periodic.

In case the motion is not periodic every point on the table lies in the path of the ball or is a limit point of such points. This can be proved by means of the following lemma: *Given any two irrational numbers a and b and any positive number e, however small, then it is possible* to find a pair of integers A and B such that* $|Aa+Bb| < e$.

Consider now the case where the ball started from A. According to the lemma for any preassigned positive number e there exists a pair of integers h and k such that $|km\tan\alpha - hn| < e$. This shows that on the extended table the ball comes as near as we please to multiple points of n. If k is even, D is obviously a limit point of the set of points passed over by the ball. In the same manner we can show that any point on either one of the sides AD or BC

*Jules Tannery, *Theorie des Fonctions*, 2d ed., I, p. 37.

which is not reached by the ball is a limit point of points which it does reach. Since the paths of the ball from one side to the opposite side are parallel it follows that every point on the table not reached by the ball is a limit point of points which it does reach.

Evidently the same conclusion follows in the non-periodic cases in which the ball starts from a point other than a corner of the table.

DEPARTMENTS.

NOTE. All solutions of problems, problems for solution, and other department contributions should be sent direct to THE AMERICAN MATHEMATICAL MONTHLY, 1227 Clay Street, Springfield, Mo.

SOLUTIONS OF PROBLEMS.

NOTE. The following solutions were received too late for publication: Calculus, No. 196, solved by J. Scheffer, and A. H. Holmes; No. 197, solved and discussed by Miss Ida M. Schottenfels. Geometry, No. 255, solved by William Hoover; No. 257, solved by F. R. Honey. Mechanics, No. 179, solved by William Hoover, and J. Scheffer.

ALGEBRA.

234. Proposed by G. W. GREENWOOD, M. A. (Oxon), Professor of Mathematics and Astronomy, McKendree College, Lebanon, Ill.

Prove that $(x+n)^n-n(x+n-1)^n+\frac{n(n-1)}{2!}(x+n-2)^n-+\ldots\ldots\ldots=n!$

Solution by the PROPOSER.

$\triangle u_x=u_{x+1}-u_x=(E-1)u_x$ where $Eu_x=u_{x+1}$.

Hence $(E-1)x^n=\triangle x^n$, $(E-1)^n x^n=\triangle^n x^n$, *i. e.*, $(x+n)^n-n(x+n-1)^n+\frac{n(n-1)}{2!}(x+n-2)^n-+\ldots\ldots=\triangle^n x^n!=n!$ by a well known theorem in the Calculus of Finite Differences.*

Also solved by J. Scheffer.

235. Proposed by WILLIAM HOOVER, Ph. D., Athens, Ohio.

Easter Sunday, 1905, was on April 23. How often in the last one hundred years has this occurred, and when?†

Solution by the PROPOSER.

The Dominical Letter for April 23, 1905, is A. No other year thus far in the twentieth century had A for its Dominical Letter.

*Epsteen, MONTHLY, June-July, 1904, p. 132.

†Dr. Zerr adds the remark that this will occur again in 1916 and 2000.

The years of the nineteenth century having A for their Dominical Letter were: 1809, 1815, 1820, 1826, 1837, 1843, 1848, 1854, 1865, 1871, 1876, 1882, 1893, the Epacts for which are 14, 20, 15, 22, 23, 0, 25, 1, 3, 9, 4, 11, 12.

Of these the years admissible must have the Epacts 24, 25, 26, 27, one only meeting the conditions, viz., 1848.

Also solved by J. Scheffer, and A. H. Holmes.

236. Proposed by L. SHIVELY, Mt. Morris College, Mt. Morris, Ill.

Sum to infinity the series $\frac{n^2}{(n+1)\,!}$ beginning with $n=1$.

Solution by L. E. NEWCOMB, Los Gatos, Cal.

The general term $\frac{n^2}{(n+1)\,!}=n\left(\frac{1}{n!}-\frac{1}{(n+1)\,!}\right)$. Set $n=1, 2, 3, \ldots$ in succession, and the series becomes

$$1-\frac{1}{2}+\frac{2}{2!}-\frac{2}{3!}+\frac{3}{3!}-\frac{3}{4!}\ldots\ldots=1+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\ldots\ldots=e-1.$$

Also solved by G. W. Greenwood, J. Scheffer, and the Proposer.

AVERAGE AND PROBABILITY.

162. Proposed by F. P. MATZ, Ph. D., Sc. D., Reading, Pa.

Two points are taken at random in the surface of a circle and a chord is drawn through them. Find the average area of the segment containing the center of the circle.

Solution by the PROPOSER.

Let 2θ=the sectoral angle, then the area of the segment in question is $U=\frac{1}{2}(2\pi-2\theta+\frac{1}{2}\sin 2\theta)r^2$.

$$\therefore A=\frac{r^2}{\pi}\int_0^{\frac{1}{2}\pi}U d\theta=\tfrac{1}{2}\left(\frac{3\,\pi}{2}+\frac{1}{\pi}\right)r^2.$$

CALCULUS.

199. Proposed by F. P. MATZ, Sc. D., Ph. D., Reading, Pa.

If the perimeter of an ellipse varies uniformly at the rate of $\frac{1}{4}$ inch per unit of time, at what rate is the eccentricity varying the instant the perimeter becomes 60 inches and the major axis 25 inches?

No correct solution for this problem has been received.

200. Proposed by R. D. CARMICHAEL, Hartselle, Alabama.

Find the equation of a curve so that the area bounded by the curve, the axis of x, and any ordinate y, is equal to $y-x$, x being the *corresponding* abscissa.

Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

$\int_0^z y\,dx = y - x$. $\therefore y\,dx = dy - dx$, or $dx = \frac{dy}{y+1}$.

$x = \log(y+1)$, and $e^x = (y+1)$ is the required equation.

Also solved by S. A. Corey, F. P. Matz, and the Proposer.

DIOPHANTINE ANALYSIS.

127. Proposed by F. P. MATZ, Sc. D., Ph. D., Reading, Pa.

Can there be determined three cube numbers whose sum is the product of two squares?

I. Solution by A. H. HOLMES, Brunswick, Maine.

Take x^3, x^3y^3, and x^3z^3, for the three cube numbers, and u^2 and v^2 for the square numbers.

$\therefore x^3(1+y^3+z^3) = u^2v^2$. If we put $y=2$, $z=3$, $1+2^3+3^3 = 6^2 = u^2$, $x^3 = v^2 = w^6$. $\therefore x = w^2$, $v = w^3$. Put $w=2$. $\therefore x=4$, $v=8$.

$\therefore 64+512+1728 = 36\times 64 = 2304$, or $4^3+8^3+12^3 = 6^2\times 8^2$.

If it is required that no two of the numbers to be cubed and squared should be the same, put $x^3 = uv$, so that $1+y^3+z^3 = uv$.

Put $x=9$, $y=6$, and $z=8$. Then $u=3$, $v=243$.

$\therefore 9^3+54^3+72^3 = 3^2\times 243^2$.

II. Solution by J. EDWARD SANDERS, Hackney, Ohio.

If $a^3+b^3+c^3 = d^3 = (e.f)^3$, then one set of numbers satisfying the condition is: $(a.d^{2n-1})^3+(b.d^{2n-1})^3+(c.d^{2n-1})^3 = d^{6n} = (e^{3n})^2.(f^{3n})^2$.

Again, since $1^3+2^3+3^3 = 2^2.3^2$, another solution is $(a^4)^3+(2a^4)^3+(3a^4)^3 = (2a^3)^2.(3a^3)^2$.

Also solved by G. B. M. Zerr, R. D. Carmichael, and the Proposer.

GEOMETRY.

NOTE. Problems 259 and 261 are identical. A solution may be found in Lachlan's *Modern Pure Geometry*, pp. 241-2.

254. Proposed by W. J. GREENSTREET, M. A., Editor of The Mathematical Gazette, Stroud, England.

Find the cartesian equation of a line that is both tangent and normal to the cardioid.

Solution by F. H. SAFFORD, Ph. D., The University of Pennsylvania.

Let the required line be $2ax+2by+c=0$, which is to be both tangent and normal to $(x^2+y^2+cx)^2 = c^2(x^2+y^2)$.

The constants a and b are to be found so as to satisfy the given conditions.

Transform the line and cardioid through the inversion

$$x=\frac{cx'}{x'^2+y'^2}, \quad y=\frac{cy'}{x'^2+y'^2}$$

obtaining, after dropping accents,

$$x^2+y^2+2ax+2by=0\,;\; y^2=2x+1.$$

Since angles are not changed by inversion the conditions to be satisfied are now that the circle shall be both tangent and normal to the parabola.

The latter condition requires that the coördinates of one intersection shall satisfy $y^2+by-x-a=0$.

Elimination of y^2 from these three equations followed by elimination of y from the two equations thus obtained gives $(x+1)(x+2a-1)=0$.

The first root is not used as it leads to an imaginary intersection.

From the second root the intersection is

$$[1-2a,\ \pm\sqrt{(3-4a)}],\ \text{also}\ b=\frac{3a-2}{\pm\sqrt{(3-4a)}}.$$

With this value of b the abscissae of intersections of circle and parabola are given by

$$(x^2+2ax+2x+1)^2-\frac{4(3a-2)}{3-4a}(2x+1)=0.$$

Removing the factor $x+2a+1$ which corresponds to the orthogonal intersection, the result is the following cubic,

$$x^3+x^2(2a+5)+11x+\frac{13-18a}{3-4a}=0.$$

As the circle is to be tangent to the parabola this equation must have a double root, *i. e.*, its discriminant must vanish, thus giving the following equation for the determination of a,

$$9a^5+24a^4-143a^3+202a^2-116a+24=0.$$

The roots of the latter are $1,\ 1,\ \frac{2}{3},\ \frac{2}{3},\ -6$, of which the first four lead to imaginary intersections.

When a is -6, b is $\frac{\mp 20\sqrt{3}}{9}$, so that the desired result is*

$$108x\pm 40\sqrt{3}y-9c=0.$$

*Dr. Hoover obtains the obvious line $y=0$, but not the ones of the published solution. Dr. Safford's fortunate choice of c for absolute term in the assumed line excludes the line $y=0$ from his result. A perfectly general method would furnish a singly infinite family of lines only the above three of which are real. An investigation of this family for, say the general nodal-circular quartic, ought to lead to some interesting results. Ed. G.

262. Proposed by NELSON L. RORAY, Utica, New York.

In a regular pentagon, show that $\frac{\text{diagonal}}{\text{side}} = \frac{2 \text{ apothem}}{\text{radius}} = \frac{1}{2}\cdot\frac{5+\sqrt{5}}{\sqrt{5}}$.

Solution by P. S. BERG, Larimore, N. Dak.

In a regular pentagon the diagonal is $\frac{R}{2}\sqrt{(10+2\sqrt{5})}$, and the side is $\frac{R}{2}\sqrt{(10-2\sqrt{5})}$.

$$\frac{\frac{R}{2}\sqrt{(10+2\sqrt{5})}}{\frac{R}{2}\sqrt{(10-2\sqrt{5})}} = \frac{\frac{R}{2}\sqrt{(6+2\sqrt{5})}}{R}.$$

But $\frac{R}{2}\sqrt{(6+2\sqrt{5})}$=2 apothem, and R=radius.

$$\frac{\frac{R}{2}\sqrt{(6+2\sqrt{5})}}{R} = \frac{\sqrt{(6+2\sqrt{5})}}{2} = \frac{1}{2}\cdot\frac{\sqrt{(30+10\sqrt{5})}}{\sqrt{5}} = \frac{1}{2}\cdot\frac{5+\sqrt{5}}{\sqrt{5}}.$$

Also solved by G. B. M. Zerr, R. D. Carmichael, A. H. Holmes, and L. E. Newcomb.

MECHANICS.

180. Proposed by EDWIN L. RICH, Lehigh University, South Bethlehem, Pa.

If a body is projected into the air, and the resistance of the air varies as the square of the velocity; required the equation of the curve. [From De Volson Wood's *Analytical Mechanics*, problem 10, p. 179.]

Solution by G. B. M. ZERR.

Let the direction of projection be in the xy plane. Let v=velocity of projection, g=acceleration of gravity, α=angle of projection, $k(ds/dt)^2$=resistance at any time t, ϕ=inclination of direction of motion to the horizon at any time t. The x- and y- components of resistance are, respectively, $k\frac{ds}{dt}\frac{dx}{dt}$, and $k\frac{ds}{dt}\frac{dy}{dt}$.

Resolving horizontally and vertically, the equations of motion are

$$d^2x/dt^2 = -k(ds/dt)(dx/dt) \ldots\ldots(1),$$
$$dy^2/dt^2 = -g-k(ds/dt)(dy/dt) \ldots\ldots(2).$$

From (1), $d(dx/dt)/(dx/dt) = -kds$.
$\therefore \log[(dx/dt)/v\cos\alpha] = -ks$. When $t=0$, $dx/dt = v\cos\alpha$.
$\therefore dx/dt = v\cos\alpha e^{-ks} = u \ldots\ldots(3)$.
Resolving in the direction of the tangent and normal,

$$d^2s/dt^2 = -g\sin\varphi - k(ds/dt)^2 \ldots\ldots(4).$$

Let v_1 = velocity of particle at any point; then (4) becomes

$$d^2s/dt^2 = -g\sin\phi - kv_1^2 \ldots\ldots(5),$$
$$v_1^2/\rho = g\cos\phi \ldots\ldots(6).$$

But $u = v_1\cos\varphi$ and $\rho = -ds/d\phi$. $\therefore$ (5) and (6) become

$$du/dt = -kv_1^2\cos\phi \ldots\ldots(7),$$
$$v_1(d\phi/dt) = -g\cos\phi \ldots\ldots(8).$$

$$\therefore \frac{du}{d\phi} = \frac{kv_1^3}{g} = \frac{k}{g}u^3\sec^3\phi \ldots\ldots(9).$$

At the origin, $u = v\cos\alpha$. Integrating (9) we get

$$\frac{1}{v^2\cos^2\alpha} - \frac{1}{u^2} = \frac{k}{g}(A_\phi - A_\alpha) \ldots\ldots(10).$$

Where $A_\phi = \int_0^\phi 2\sec^3\phi d\phi$, $A_\alpha - A_\phi = 2\int_\phi^\alpha \sec^3\phi d\phi$

$$= \tan\alpha\sec\alpha - \tan\phi\sec\phi + \log\left(\frac{\tan\alpha + \sec\alpha}{\tan\phi + \sec\phi}\right).$$

Substituting in (10) we get

$$\frac{k}{g}v^2\cos^2\alpha\left[\tan\alpha\sec\alpha - \tan\phi\sec\phi + \log\left(\frac{\tan\alpha+\sec\alpha}{\tan\phi+\sec\phi}\right)\right] = e^{2ks} - 1,$$

for the intrinsic equation to the curve.

MISCELLANEOUS.

148. Proposed by F. P. MATZ, Ph. D., Sc. D., Reading, Pa.

Given $\sin 3\phi + \cos 3\phi = m$.....(1), and $\cos\phi - \sin\phi = x$.....(2), to find x in terms of m.

Solution by J. SCHEFFER, A. M., Hagerstown, Md.

Since $3\sin\phi - 4\sin^3\phi = \sin 3\phi$ and $-3\cos\phi + 4\cos^3\phi = \cos 3\phi$, we have $4(\cos^3\phi - \sin^3\phi) - 3(\cos\phi - \sin\phi) = m$, or $(\cos\phi - \sin\phi)[4(\cos^2\phi + \cos\phi\sin\phi + \sin^2\phi) - 3] = m$.

Since $\cos\phi - \sin\phi = x$, $2\sin\phi\cos\phi = 1 - x^2$ and therefore $2x^3 - 3x + m = 0$.

Also solved by A. H. Holmes, and the Proposer.

PROBLEMS FOR SOLUTION.

ALGEBRA.

240. Proposed by F. P. MATZ, Sc. D., Ph. D., Reading, Pa.

Solve $a^2x+b^2y=ax^2+by^2=x^3+y^3$.

241. Proposed by L. E. NEWCOMB, Los Gatos, Cal.

Sum to infinity $\frac{1}{2}+\frac{1}{3}+\frac{1}{8}+\frac{1}{30}+\frac{1}{840}+\cdots$

242. Proposed by DR. L. E. DICKSON, The University of Chicago.

If $u_0{}^i h_0+u_1{}^i h_1+\cdots\cdots u_r{}^i h_r=0$ $(i=0, 1, \ldots\ldots, r-1)$, then $h_i \prod_{\substack{j=0,1,\ldots,r \\ j \text{ not} = \text{to } i}} (u_i-u_j)=0$.

243. Proposed by WILLIAM HOOVER, Ph. D., Athens, Ohio.

Find the infinite root of $\frac{1}{x}+\frac{1}{a}=\sqrt{\left[\frac{1}{a^2}-\sqrt{\frac{1}{a^2x^2}+\frac{1}{x^4}}\right]}$.

AVERAGE AND PROBABILITY.

168. Proposed by J. SCHEFFER, A. M., Hagerstown, Md.

Find the average area of a triangle two of whose sides have the constant sum $2a$.

169. Proposed by HENRY HEATON, Atlantic, Iowa.

What is the average length of all straight lines that can be drawn within a given square?

170. Proposed by LON C. WALKER, Santa Barbara, Cal.

Find the area of a triangle formed by drawing a line at random through each of the three points taken at random within the surface of a given triangle.

171. Proposed by O. E. GLENN, A. M., Ph. D., Drury College.

There are n derelict steamers afloat in a circular sea of radius r. The water in the sea is moving northward in a current whose velocity varies inversely as the perpendicular distance from the north-south tangent to the sea on its west beach. Find the probability that a ship crossing the sea on a random diameter will encounter e derelicts during the voyage.

CALCULUS.

202. Proposed by W. J. GREENSTREET, M. A., Editor of the Mathematical Gazette, Stroud, England.

Find the complete primitive of $y=2px+ap^2$. Regard the primitive as the equation giving the arbitrary constant, and if the primitive has equal roots discuss the equation expressing that condition.

203. Proposed by S. A. COREY, Hiteman, Iowa.

$$\int_0^\pi \frac{\sin mx}{x} dx, \; m = \text{integer}.$$

204. Proposed by M. E. GRABER, A. M., Heidelberg University, Tiffin, Ohio.

Required the variation of $\int V dx$ where V is a function of x, y, $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$ ..... and v where $v = \int V' dx$ and V' is also a function of x, y, $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$ .....

205. Proposed by Z. T. JACKSON, St. Louis, Mo.

Evaluate $\int_0^{\frac{1}{2}\pi} \log \sin x \, dx$.

206. Proposed by DR. O. E. GLENN, Drury College.

Evaluate $\int_0^1 (1-z^n)^m \frac{\partial}{\partial z} \log(1-z^n x^n) dz$, assuming $-1 < x^n < +1$.

DIOPHANTINE ANALYSIS.

128. Prpposed by F. P. MATZ, Ph. D., Sc. D., Reading, Pa.

Required the highest powers of 2, 3, 5, 7, contained in (1000)!

129. Proposed by SYLVESTER ROBINS.

How many perfect squares containing 2^n figures each can be found, the parts of which standing on the right hand side thereof, represented by 1, 2, 4, 8, 16, 32, etc., digits, are also perfect squares. 24591681 is one such number. [From *The Mathematical Visitor*].

GEOMETRY.

263. Proposed by FREDERICK R. HONEY, Trinity College, Hartford, Conn.

Construct a sphere whose surface shall intersect the surface of any four given spheres in great circles.

264. Proposed by B. F. FINKEL, A. M., Drury College, Springfield, Mo.

Let l and m be two straight lines intersecting in A. With A as center and any radius r describe a circle intersecting l and m in E, M and G, Q, respectively, and the bisector of the opposite angles formed by l and m in F and K. With I, the middle point of EA, as center, and radius, r, describe an arc intersecting the bisector of the opposite angles formed by l and m in O. With O as center, and radius $OA+r$ describe circle $FHCDBJF$, F and D the points of intersection of this circle with the bisector of opposite angle; H, B the intersections on l, and J, C on m. What is the ratio of arc HFJ to arc BD?

265. Proposed by R. D. CARMICHAEL, Hartselle, Ala.

Find the Cartesian equation of a curve in a vertical plane such that a particle, sliding down the curve under the force of gravity alone, will require to pass from any point of beginning to the lowest point of the curve, a time proportional to the square of the distance to be traversed along the curve.

266. Proposed by DR. O. E. GLENN.

Given the feet of the three perpendiculars from any point a on the circumcircle to the sides of the triangle are collinear, then if on the three chords $\overline{a\,b}$, $\overline{a\,c}$, $\overline{a\,d}$, as diameters circles be described, the points of intersection of these circles are collinear. [Salmon's *Higher Plane Curves*].

267. Proposed by W. W. LANDIS, Dickinson College, Carlisle, Pa.

Prove that every orthogonal system of circles in a plane is an isothermal system.

GROUP THEORY.

9. Proposed by DR. L. E. DICKSON, The University of Chicago.

Does there exist a triply transitive group on m letters of order $m(m-1)(m-2)$ other than the linear fractional group in the Galois Field of order $p^n=m-1$ and the group 720_3 on 10 letters (Cole, *Quarterly Journal*, 1895, p. 44)? The question relates to Problem 99, MONTHLY, March, 1900.

10. Proposed by DR. O. E. GLENN.

Find the order of the group of isomorphisms of the group of order p^4 defined by the relations $P_1^{p^2}=P_2^{p^2}=I$, $P_1P_2=P_2P_1$.

MECHANICS.

183. Proposed by W. J. GREENSTREET, M. A., Stroud, England.

Two light rods are jointed at O. OA is attached by a hinge at A to a fixed rod AB and B is attached to a ring which can slide along AD. A force P acts at O towards AB at right angles to AB, and force Q acts at B along BA. The angles OAB, OBA are acute. There is no friction in the system. Show that for equilibrium we must have $P/Q=\sin AOB/(\cos AOB.\cos OBA)$.

184. Proposed by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

A sphere, radius a, rests between two parallel thin perfectly rough rods A and B in the same horizontal plane at a distance apart equal to $2c$; the sphere is turned about A until its center is very nearly vertically over A; it is then allowed to fall back. Prove that it will rock between A and B if $10c^2<7a^2$; also, that θ_r, the angle through which it will turn after the rth impact is given by the equation $\cos\theta_r=\frac{\sqrt{(a^2-c^2)}}{a}+\frac{a-\sqrt{(a^2-c^2)}}{a}\left(1-\frac{10c^2}{7a^2}\right)^{2r}$.

MISCELLANEOUS.

150. Proposed by T. N. HAUN, Mohawk, Tenn.

If $\frac{\sin\phi}{\sin\psi}=m$, find maximum and minimum value of $\frac{\sin(\phi+\theta)}{\sin(\psi+\theta)}$, where θ is known.

151. Proposed by W. J. GREENSTREET.

Sum the series $\sum_{r=1}^{r=m} \operatorname{cosec}\left(\frac{2r-1}{4m}\pi+\theta\right)\operatorname{cosec}\left(\frac{2r-1}{4m}\pi-\theta\right)$.

NOTES.

Mr. A. R. Crathorne of the University of Wisconsin is a student at Göttingen.

Professor W. D. Cairns of Oberlin College is spending the year in research at Göttingen.

Professor H. S. White of Northwestern University has been called to Vassar College.

Mr. W. C. Brenke has been appointed Austin Teaching Fellow in Astronomy at Harvard.

Mr. T. C. Jones has been appointed fellow in mathematics at the University of Pennsylvania.

Mr. H. W. Reddick has been appointed fellow in mathematics at the University of Illinois.

Dr. W. D. Westfall has been appointed instructor in mathematics at the University of Missouri.

Mr. Homer R. Higley has been appointed instructor in mathematics in the Pennsylvania State College.

Dr. W. B. Ford of Williams College has been appointed instructor in mathematics at the University of Michigan.

The assistants in mathematics for 1905-6 at Indiana University are Mr. Charles Haseman, and Mr. D. J. Crittenberger.

The University of Washington has appointed Professor F. M. Morrison of Buchtel College as assistant professor of mathematics.

Professor H. F. Blichfeldt will spend next year in research at European universities, on leave of absence from Stanford University.

Professor G. A. Bliss, of the University of Missouri, has been appointed assistant professor of mathematics at Princeton University.

At the University of Indiana Dr. S. C. Davisson, and Dr. David A. Rothrock have been promoted to junior professorships in mathematics.

Dr. L. I. Neikirk, formerly Harrison Research Fellow at the University of Pennsylvania, has been appointed instructor in mathematics at the University of Illinois.

Mr. Louis Fussell, Lippincott Fellow in Mathematical Physics at Swarthmore College, is spending the year in graduate study at the University of Wisconsin.

Professor B. F. Finkel has been appointed Harrison Fellow in mathematics at the University of Pennsylvania, and is spending the year in graduate study in Philadelphia.

The Springfield Section of the Missouri Society of Teachers of Mathematics has effected organization for the year with a membership of twenty-five. The executive committee in charge of the work consists of Dr. O. E. Glenn, Miss Elizabeth Park, and Miss Nena Baxter.

The July number of the *Bulletin of the American Mathematical Society* contains Professor Henry Dallas Thompson's translation of M. Gaston Darboux's address, "A Survey of the Development of Geometric Methods." The address was delivered at St. Louis, September 24, 1904, before the geometry section of the International Congress of Arts and Science.

The Bolyai prize, established by the Hungarian Academy of Sciences in memory of John Bolyai and his father Farkas Bolyai, will be awarded this year for the first time. The awarding commission will deliberate in Budapest in October. It consists of the following members: Gaston Darboux (Paris), Felix Klein (Göttingen), Julius König (Budapest), Gustav Rados (Budapest).

The prize consists of a medal and ten thousand crowns, and will be awarded to the author of the best work in mathematics published during the five years preceding.

The current number of *Proceedings of the London Mathematical Society* contains the following papers: On the Reducibility of Covariants of Binary Quantics of Infinite Order, by Mr. P. W. Wood; Alternative Expressions for Perpetuant Type Forms, by Mr. P. W. Wood; Theorems on the Logarithmic Potential, by Prof. T. J. I'A. Bromwich; Ordinary Inner Limiting Sets in the Plane or Higher Space, by Dr. W. H. Young; A Method for Determining the Behaviour of Certain Classes of Power Series Near a Singular Point of the Circle of Convergence, by Mr. G. H. Hardy; The Intersection of Two Conic Sections, by Mr. J. A. H. Johnston.

In *Science* of August 18 Professor G. A. Miller states that at the celebration of the last birthday of the Emperor of Germany Professor Harzer delivered a long address on the 'Exact Sciences in Old Japan.' Although Professor Harzer is an astronomer, he devoted nearly his entire address to the history of mathematics, saying that the 2,000 mathematical works in the royal library of Tokio, some of which date back to 1595, are a sufficient guarantee of high esteem for mathematical knowledge. As the Japanese mind is very practical, it is to be expected that their mathematical achievements are in very close touch with practical problems and are foreign to those fields of mathematics which border on philosophy. The determination of the area of the circle in terms of its diameter is one of the most important of these practical problems and the Japanese have taken especial interest in developments which are useful to obtain an approximate solution of this problem.

The most surprising fact about Japanese mathematics is that, formerly, while the most elementary parts were regarded as common property, the more advanced results were regarded as secrets which should be communicated to a very few. In fact, an oath of secrecy was required of those who wished to hear lectures on advanced mathematics. European history furnishes a parallel to this in the Pythagorean school, but it is so totally different from the modern spirit that its existence 2,000 years after Pythagoras was unexpected. Fortunately all this has been recently changed to such an extent that a history of Japanese mathematics could be published a few years ago. This is by Tsuruichi Hayashi and a chapter of it entitled "A Brief History of Mathematics in Japan," has been translated into English and published in *Nieuw Archief Voor Wiskunde*, 1904, pp. 296-324, and 1905, pp. 325-361.

We learn from *Science* that the conference held at Asbury Park on July 5, 1905, for the purpose of discussing the advisability of organizing a national society of teachers of mathematics, was attended by a large delegation. After some discussion a resolution was adopted to the effect that a national society of teachers of mathematics and science be organized, and the details of the organization were referred to the following executive committee: Professor Thomas S. Fiske (chairman), New York; Professor C. E. Comstock, Peoria, Ill.; Professor E. R. Hedrick, Columbia, Mo.; Mr. Franklin T. Jones, Cleveland, O.; Professor William H. Metzler, Syracuse, N. Y.; Mr. Edgar H. Nichols, Cambridge, Mass.

A report of the proceedings of this committee will be published in *School Science and Mathematics*.

Following is a list of the associations represented at the Asbury convention, with names of delegates representing each association.

New England Mathematics Teachers Association.—Chas. E. Bouton, Harvard University; Paul Capron; Mr. Nichols, Brown and Nichols School, Cambridge.

Association of Teachers of Mathematics in the Middle States and Maryland.—John C. Bechtel; Fletcher Durell, Lawrenceville, N. J.; A. Newton Ebaugh; Miss Susan C. Lodge; Donald C. MacLaren; Wm. H. Metzler, Syracuse Univer-

sity; J. T. Rorer, Central High School, Philadelphia; Arthur Schultze, High School of Commerce, New York; H. C. Whitaker.

Central Association of Science and Mathematics Teachers.—Otis W. Caldwell; Jos. V. Collins; C. E. Comstock; G. W. Greenwood; Charles H. Smith; Charles M. Turton; J. W. Young, Charles W. Wright.

Missouri Society of Teachers of Mathematics.—F. T. Appleby; J. S. Bryan, Central High School, St. Louis; H. Clay Harvey; E. R. Hedrick; B. F. Johnston; John R. Kirk; J. W. Whiteye.

Chicago and Cook County High School Teachers' Association.—Edward E. Hill; Fred R. Nichols; Charles M. Turton.

Mathematical Section of Michigan School-Masters' Club.—Miss Emma C. Ackermann.

New York State Science Association, Mathematical Department.—Glenn M. Lee.

North Eastern Ohio Center, G. A. S. and M. T.—Lemar T. Beman, Cleveland High School; Charles A. Marple.

Ohio Association of Teachers of Mathematics and Science.—Franklin T. Jones; Wm. McLair.

St. Louis Association of Science and Mathematics Teachers.—Wm. Schuyler, McKinley High School, St. Louis.

BOOKS.

A College Algebra. By Henry Burchard Fine, Professor of Mathematics in Princeton University. viii+595 pages, Cloth. Boston, New York, Chicago, and London: Ginn & Co.

Those who have read Professor Fine's *The Number System of Algebra* will expectantly turn, in this volume, to part first, entitled, Numbers. Their anticipations will be fully met. Under the influence of the Cantor-Dedekind school of thought recent years have shown remarkable advancement of the theory of irrationals and of cardinal number. The elements of this theory lie at the basis of algebra and we have here, we believe, for the first time, a teachable treatise based on these fundamental concepts.

The treatment of the usual subject-matter of college algebra is also admirable. Special features of the text are, short chapters on Interpolation, on Variation, and on Probability, and a moderately long treatment of the elements of the theory of equations. The work ends with a rigid proof of the existence theorem of algebra.

The author has given little space to special devices but seeks "to assist the student to really master the general methods of the science," and the logical interrelations which connect its parts. The work is in every sense a classic. G.

Elements of Descriptive Geometry. By Chas. E. Ferris, Professor of Mechanical Engineering in the University of Tennessee. Cloth, 8vo, 135 pages. New York, Cincinnati, Chicago: American Book Co.

The author follows the custom of most draftsmen and does all his work in the third

quadrant. The exercises are well graded and the illustrative drawings are plentiful and well executed. The first six chapters treat the usual subject matter of the subject. Chapter VII is on Warped Surfaces; Chapter VIII treats the subject of Shades and Shadows, and Chapter IX is devoted to a brief treatment of Perspective. G.

The Elements of Geometry. By Walter N. Bush and John B. Clarke, of the Polytechnic High School of San Francisco. 355 pages. Price, $1.25. Chicago, Boston, New York: Silver, Burdett & Co.

On page 255 of his book entitled "The Teaching of Elementary Mathematics" Professor David Eugene Smith of Columbia University gives the following criteria for a usable text-book in Geometry, viz:

1st. A sequence of propositions which is not only logical, but psychological;

2nd. Exact statements;

3rd. Proofs which are models of excellence for the pupil to pattern after;

4th. Abundant exercises from the beginning with practical suggestions to methods of attacking them.

Judged by this standard "The Elements of Geometry" by Bush and Clarke should prove a very successful book. The sequence of propositions is a logical one; cognate theorems are put in special groups or classes. The merit of the arrangement from the psychological point of view has been tested by the authors during the many years they have devoted to the preparation of the work. The statements and definitions are accurate and clear. In the proofs the hypothesis and conclusion are carefully separated, thus not only making the path easier for the student, but also giving him a standard for the presentation of his solution of "originals." The exercises are both numerous and well selected.

The authors are to be commended for arranging the proofs so that it should not be necessary to turn the page and then have to turn back for the figure. As in all modern texts, the diagrams are carefully prepared and are very good. An excellent index at the end enables one to find easily any desired place. ALMA E. KLUNDER.

Biology and Mathematics. By George Bruce Halsted. 15 pages. An address delivered before Section A of the American Association for the Advancement of Science, Philadelphia, December 29, 1904. Reprinted from *Science*, N. S., Vol. XXII, No. 554, pages 161-167, August 11, 1905.

Professor Halsted discusses with great force and insight the analogies between the Darwinian theory of variation and the continuous variable, on the one hand, and the more recent theory of mutation and discontinuous variation, on the other. F.

ERRATA.

Equation (3) on page 122 should read,

$$x=\frac{p}{3c}\left[\frac{F'(0)+F'(p)}{2}+F'\left(\frac{p}{3}\right)+F'\left(\frac{2p}{3}\right)\right]-\frac{p^2}{108c}\left[F''(p)-F''(0)\right] \quad \ldots\ldots (3).$$

Line 10 from bottom, on page 134, should read,

$$\begin{aligned} 2S=&\frac{1}{2.3.4}+\frac{1}{5.6.7}+\frac{1}{8.9.10}+\ldots\ldots\\ =&\tfrac{1}{2}-\tfrac{2}{3}+\tfrac{1}{4}\\ &+\tfrac{1}{3}-\tfrac{2}{4}+\tfrac{1}{5}\\ &+\tfrac{1}{4}-\tfrac{2}{5}+\tfrac{1}{6}\\ &\vdots\quad\vdots\quad\vdots\\ =&\tfrac{1}{2}-\tfrac{1}{3}=\tfrac{1}{6}. \end{aligned}$$

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THE AMERICAN MATHEMATICAL MONTHLY.

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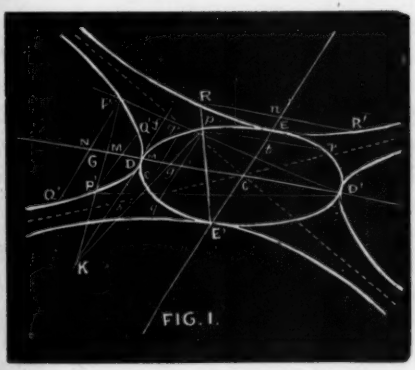
SOME NEW METRICAL PROPERTIES OF CONIC CURVES.

By REV. ALAN S. HAWKESWORTH.

Theorem 1. If through the extremities of a double ordinate to any diameter of an ellipse right lines be drawn from the extremities of said diameter; then such lines will ever intersect upon a hyperbola, having said diameter and its conjugate in common with the ellipse; and thus asymptotes which are both parallel to and bisected by the supplementary chords uniting said conjugate diameters. While lastly, the double ordinate of the said point on the hyperbola, parallel to the given double ordinate in the ellipse, will cut, with it, their common bisecting diameter harmonically.

And conversely, of course, lines through the extremities of any double ordinate in a hyperbola from the ends of its diameter, will meet upon an ellipse having said diameter and its conjugate in common; and the ordinates of the corresponding points will cut their common diameter harmonically.

For let pTq be a double ordinate in an ellipse [Fig. 1], cutting its bisecting diameter DCD' in T; and let $D'pP$ and qDP meet in P. Draw PNQ parallel to pTq, meeting DCD' in N, and pD in Q. And draw ECE' the conjugate diameter to DCD'.



Then qTD and PND being similar triangles; and likewise pTD' and PND'; while $qT=pT$, and $PN=QN$; therefore $pT : DT=qT : DT=PN : DN$, while $pT : D'T=PN : D'N$. So that $pT^2 : DT.D'T=PN^2 : DN.D'N=QN^2 : DN.D'N=CE^2=CD^2$; which latter ratio belongs to both ellipse and hyperbola. And thus P and Q must ever lie on a hyperbola, having DCD' and ECE' for its conjugate diameters also.